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NONSTATIONARY DIFFRACTION OF ELASTIC WAVES ON A RIGID ELLIPTICAL CYLINDER

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Extensive literature has been devoted to the problems of diffraction of elastic and acoustic waves by inclusions with circular cross-sections (see reviews [1–3]). The problems of the interaction of elastic waves with extended inclusions having elliptical cross-sections have been less investigated. One approach to the solution of such problems is the classical approach based on the method of expansion in terms of eigenfunctions. The scattering of scalar SH-waves on an elliptical cylinder has been studied by means of expansion in terms of the Mathieu's function in [4–6]. In the case of diffraction of P- and SV-waves, the vector wave equation is not split in elliptical coordinates by using the Mathieu's functions because of the existence of two different speeds of elastic waves, as was noted in [3, 7, 8]. Therefore, other methods for solving the problems with P- and SV-waves are employed in the majority of works [7–16]. For example, the solution is constructed by the method of scattering matrices [7, 8] or by using the method of finite-difference approximation of contour integrals [9]. The solution is sought in the form of the Papkovich–Neuber potentials [10], and the methods of the theory of complex variable functions [11] and the method of neighboring characteristics are employed [12, 13, 15]. In [14], the problem is solved with the help of Mathieu's functions. An infinite system of equations for determining an infinite number of unknown coefficients is obtained which is solved numerically.

An analysis of the literature shows that the major part of the works is devoted to the investigation of diffraction of harmonic and stationary waves by elliptical obstacles [7–14]. The nonstationary interaction of longitudinal waves with an elliptical cavity has been studied with the help of Debye radial series in [16], where numerical results obtained by using a simplified computation scheme are presented. No detailed investigation of the nonstationary diffraction of plane P- and SV-waves on elliptical obstacles has been carried out. Also, simple analytical estimates of the solution of such problems are lacking.

This paper proposes an approximate approach to the separation of variables in the equations of linear elasticity theory for a problem with an elliptical boundary under the action of plane elastic waves. The approach is demonstrated in the solution of nonstationary problems of diffraction of elastic waves by a rigid inclusion. The asymptotic stress values ($t \to \infty$, t is time) at the cylinder surface are found approximately. It is shown that in the particular case of a round cylinder this approach leads to the exact solution of the problem.

Statement of the Problem. The problem on the influence of plane P- and SV-waves on an infinitely long rigid cylinder surrounded by an elastic medium is investigated. The cylinder has an elliptical cross section. The plane statement of the problem is considered: the front of the incident wave is parallel to the cylinder axis. The direction of movement of the incident wave makes an angle θ with the major axis of the ellipse (Fig. 1). In the (x', y') coordinate system rotated through an angle $\theta - \pi/2$ to the (x, y) coordinates, the stresses in the incident wave are given in the following way:

$$\sigma^0_{y'y'} = -\sigma_1 H_0(z_1), \quad \sigma^0_{x'x'} = \sigma_1 \varepsilon H_0(z_1), \quad \sigma^0_{x'y'} = -\sigma_2 H_0(z_2), \quad z_i = c_i t - a_1 + y', \quad \varepsilon = -\nu/(1-\nu), \qquad (1)$$

where H_0 is a Heaviside unit stepfunction; σ_1, σ_2 are the stresses at the front of the incident longitudinal and transverse waves, respectively; the incident wave propagates in the y' direction; ν is the Poisson coefficient;

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c_1 is the speed of expansion waves; c_2 is the speed of shear waves; and a_1 is the major semi-axis of the ellipse. Let us introduce an elliptical coordinate system associated with the cylinder:

$$x = a\cosh\zeta\cos\eta, \quad y = a\sinh\zeta\sin\eta \quad (0 \leqslant \zeta < \infty,\ 0 \leqslant \eta \leqslant 2\pi)$$

($2a$ is the distance between the two foci of the ellipse). The value $\zeta = \zeta_0$ corresponds to the cylinder surface. The movement of the elastic medium is described by two-dimensional wave equations for the scalar φ and vector ψ potentials of the displacements:

$$\frac{\partial^2\varphi}{\partial t^2} = c_1^2\Delta\varphi, \quad \frac{\partial^2\psi}{\partial t^2} = c_2^2\Delta\psi, \quad c_2^2 = æ c_1^2, \quad æ = (1+\varepsilon)/2 \tag{2}$$

(Δ is the Laplacian). The potentials φ and ψ must be nonzero in the expanding region limited by the perturbation front and equal to zero outside this region.

The separation of the problem into direct and additional problems $Y^\Sigma = Y^0 + Y^1$ (Y^0 are components of stresses and displacements in the incident wave and Y^1 corresponds to the reflected and diffraction waves) is usually used in linear diffraction problems.

The conditions of absence of displacements are set at the surface of the rigid cylinder

$$u_\zeta^0 + u_\zeta^1 = 0, \quad u_\eta^0 + u_\eta^1 = 0 \quad (\zeta = \zeta_0) \tag{3}$$

(u_ζ, u_η are the normal and tangent displacements). At $t = 0$, we have zero initial conditions.

To solve the problem, we apply Laplace's transform in time with parameter p to Eq. (2) and to the boundary conditions (3):

$$p^2\varphi^{1L} = c_1^2\Delta\varphi^{1L}, \qquad p^2\psi^{1L} = c_2^2\Delta\psi^{1L}; \tag{4}$$

$$u_\zeta^{0L} + u_\zeta^{1L} = 0, \qquad u_\eta^{0L} + u_\eta^{1L} = 0 \qquad (\zeta = \zeta_0). \tag{5}$$

Since Eq. (4) with conditions (5) in the elliptical coordinate system cannot be separated because of the existence of two speeds of perturbation propagation [3], it is necessary to pass from the elliptical coordinate system (ζ, η) to the cylindrical coordinate system (r, α) in accordance with the relations

$$r^2 = a^2(\sinh^2\zeta + \cos^2\eta), \quad \omega = \cosh\zeta\cos\eta\cos\theta + \sinh\zeta\sin\eta\sin\theta = ra^{-1}\cos(\theta - \alpha). \tag{6}$$

The relation between the stresses and displacements in different coordinate systems has the form

$$\begin{gathered}\sigma_{\zeta\zeta} = \sigma_{rr}\alpha_1^2 + \sigma_{\alpha\alpha}\beta_1^2 + 2\sigma_{r\alpha}\alpha_1\beta_1, \qquad \sigma_{\eta\eta} = \sigma_{rr}\alpha_2^2 + \sigma_{\alpha\alpha}\beta_2^2 + 2\sigma_{r\alpha}\alpha_2\beta_2, \\ \sigma_{\zeta\eta} = \sigma_{rr}\alpha_1\alpha_2 + \sigma_{\alpha\alpha}\beta_1\beta_2 + \sigma_{r\alpha}(\alpha_1\beta_2 + \alpha_2\beta_1), \quad u_\zeta = u_r\alpha_1 + u_\alpha\beta_1, \quad u_\eta = u_r\alpha_2 + u_\alpha\beta_2,\end{gathered} \tag{7}$$

where

$$\alpha_1 = \beta_2 = J_{,\zeta}\big/\sqrt{J_0}; \quad \beta_1 = -\alpha_2 = J_{,\eta}\big/\sqrt{J_0}; \quad J = a^2(\sinh^2\zeta + \sin^2\eta); \quad J_0 = J_{,\zeta}^2 + J_{,\eta}^2.$$

Here and below, the comma in the index means differentiation with respect to the corresponding argument. The expression of stresses and displacements in terms of the potentials φ and ψ in the cylindrical coordinate system is written as follows:

$$\begin{gathered}\sigma_{rr} = 2c_2^2\rho\big[D_1(\varphi) + D_2(\psi)\big], \qquad \sigma_{r\alpha} = 2c_2^2\rho\big[D_2(\varphi) - D_1(\psi)\big], \\ \sigma_{\alpha\alpha} = 2c_2^2\rho\big[D_3(\varphi) - D_2(\psi)\big], \qquad u_r = \varphi_{,r} + \psi_{,\alpha}\big/r, \qquad u_\alpha = \varphi_{,\alpha}\big/r - \psi_{,r},\end{gathered} \tag{8}$$

where ρ is density of the elastic medium and D_1, D_2, D_3 are operators:

$$D_1(f_i) = \frac{d_i}{2}\Delta f_i - \frac{1}{r}f_{i,r} - \frac{1}{r^2}f_{i,\alpha\alpha}, \quad D_2(f_i) = \frac{1}{r}f_{i,r\alpha} - \frac{1}{r^2}f_{i,\alpha} \quad (i = 1, 2),$$

$$D_3(\varphi) = -D_1(\varphi) + (1-\varepsilon)\Delta\varphi/2æ, \qquad d_1 = 1/æ, \qquad d_2 = 1, \qquad f_1 = \varphi, \qquad f_2 = \psi.$$

Substituting (8) into (7), for stresses and displacements in the elliptical coordinate system, we obtain

$$\sigma_{\zeta\zeta} = 2c_2^2\rho\big[B_1(\varphi) + B_2(\psi)\big]\big/J_0, \quad \sigma_{\zeta\eta} = 2c_2^2\rho\big[B_2(\varphi) - B_1(\psi)\big]\big/J_0, \quad \sigma_{\eta\eta} = 2c_2^2\rho\big[B_3(\varphi) - B_2(\psi)\big]\big/J_0, \tag{9}$$

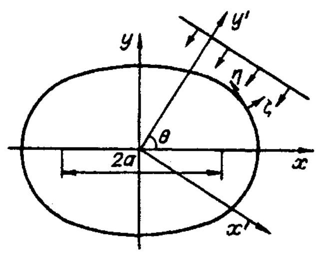

Fig. 1

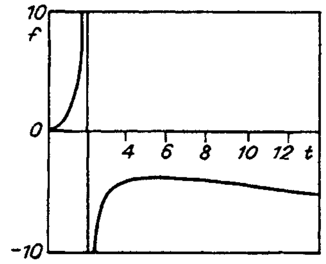

Fig. 2

$$u_\zeta = \left[J_{,\zeta}\left(\varphi_{,r} + \psi_{,\alpha}/r\right) + J_{,\eta}\left(\varphi_{,\alpha}/r - \psi_{,r}\right)\right]/\sqrt{J_0}, \quad u_\eta = \left[J_{,\zeta}\left(\varphi_{,\alpha}/r - \psi_{,r}\right) - J_{,\eta}\left(\varphi_{,r} + \psi_{,\alpha}/r\right)\right]/\sqrt{J_0}.$$

Here

$$B_1(f_i) = bB_{11}(f_i) + dB_{12}(f_i); \quad B_2(f_i) = dB_{21}(f_i) - bB_{22}(f_i); \quad B_3(\varphi) = -B_1(\varphi) + (1-\varepsilon)J_0\Delta\varphi/2æ;$$

$$B_{11}(f_i) = -\frac{b_i}{2b}\Delta f_i + \frac{1}{r}f_{i,r} + \frac{1}{r^2}f_{i,\alpha\alpha}; \qquad B_{21}(f_i) = -\frac{1}{2}\Delta f_i + \frac{1}{r}f_{i,r} + \frac{1}{r^2}f_{i,\alpha\alpha};$$

$$B_{22}(f_i) = B_{12}(f_i) = \frac{1}{r}f_{i,r\alpha} - \frac{1}{r^2}f_{i,\alpha}; \quad b = J^2_{,\eta} - J^2_{,\zeta}; \qquad d = 2J_{,\eta}J_{,\zeta}; \qquad b_1 = (\varepsilon J^2_{,\eta} - J^2_{,\zeta})/æ; \qquad b_2 = b.$$

The image of the potentials φ and ψ in the incident wave has the form

$$\varphi^{0L} = -A_1\sigma_1 W_1/\rho p^3, \quad \psi^{0L} = -A_2\sigma_2 W_2/\rho p^3, \quad W_i = \exp(pa\omega/c_i), \quad A_i = \exp(-pa_1/c_i). \tag{10}$$

We expand the functions W_i into a series in terms of modified Bessel functions [17]:

$$W_i = \sum_{n=0}^{\infty} e_n I_n(\delta_i)\cos n(\theta-\alpha), \quad e_n = 2 \quad (n>0), \quad e_0 = 1, \quad \delta_i = rp/c_i. \tag{11}$$

The solution of the wave equations (4) in the cylindrical coordinate system with allowance for the absence of radiations at infinity is represented in the form

$$\begin{aligned} \varphi^{1L} &= \sum_{n=0}^{\infty} K_n(\delta_1)\left[C_{1n}\cos n(\theta-\alpha) + S_{1n}\sin n(\theta-\alpha)\right], \\ \psi^{1L} &= \sum_{n=0}^{\infty} K_n(\delta_2)\left[C_{2n}\cos n(\theta-\alpha) + S_{2n}\sin n(\theta-\alpha)\right], \end{aligned} \tag{12}$$

where K_n are the MacDonald functions of the nth order and C_{1n}, C_{2n}, S_{1n}, S_{2n} are unknown coefficients.

In order to satisfy the boundary conditions (5), we substitute (9)–(12) into (5) and obtain a system of two linear equations for the coefficients C_{1n}, C_{2n}, S_{1n}, S_{2n} ($n = 0, \dots, \infty$). Since this system is valid for an arbitrary angle θ and the functions sin, cos are orthogonal on the interval $[0, 2\pi]$, each term at $\sin n\theta$ and $\cos n\theta$ must be equal to zero. As a result, we have, for each n, a system of four equations for the coefficients C_{1n}, C_{2n}, S_{1n}, S_{2n}:

$$\begin{gathered} \begin{cases} K_{n,1}nC_{1n} - K'_{n,2}\delta_2 S_{2n} = \gamma_{1n}I_{n,1}n, \\ K'_{n,1}\delta_1 C_{1n} - K_{n,2}nS_{2n} = \gamma_{1n}I'_{n,1}\delta_1, \end{cases} \qquad \begin{cases} K'_{n,1}\delta_1 S_{1n} + K_{n,2}nC_{2n} = \gamma_{2n}I_{n,2}n, \\ K_{n,1}nS_{1n} + K'_{n,2}\delta_2 C_{2n} = \gamma_{2n}I'_{n,2}\delta_2, \end{cases} \\ K_{n,i} = K_n(\delta_i), \quad I_{n,i} = I_n(\delta_i), \quad \gamma_{in} = A_i\sigma_i e_n/\rho p^3, \quad \delta_i = rp/c_i, \quad r = a\sqrt{\sinh^2\zeta_0 + \cos^2\eta}\,. \end{gathered} \tag{13}$$

Here the prime denotes differentiation with respect to the argument. It is seen from (13) that the coefficients C_{1n}, C_{2n}, S_{1n}, S_{2n} are functions of α. Strictly speaking, in this case formulas (12) do not give the exact solution to problem (4)–(5). However, if we neglect the derivatives of the functions C_{1n}, C_{2n}, S_{1n}, S_{2n} with respect to α, we believe that formulas (12) with the coefficients C_{1n}, C_{2n}, S_{1n}, S_{2n} determined from (13) give an approximate solution to the boundary value problem (4), (5).

Solving system (13) and substituting its solution into (12) and then into (9), and making some successive simplifications, we obtain an approximate image solution for the total stresses at the surface of the rigid elliptical cylinder:

$$
\begin{gathered}
\sigma_{\zeta\zeta}^{\Sigma L} = \sum_{n=0}^{\infty} \sigma_{\zeta\zeta,n}^{\Sigma L}, \quad \sigma_{\zeta\eta}^{\Sigma L} = \sum_{n=0}^{\infty} \sigma_{\zeta\eta,n}^{\Sigma L}, \quad \sigma_{\eta\eta}^{\Sigma L} = \sum_{n=0}^{\infty} \sigma_{\eta\eta,n}^{\Sigma L}, \\
\sigma_{\zeta\zeta,n}^{\Sigma L} = \frac{e_n}{J_0 \Omega_n p}\Big[\cos n(\theta-\alpha)\big(-\sigma_1 A_1 æ b_1 \delta_2 K'_{n,2} - \sigma_2 A_2 d \delta_1 K'_{n,1}\big) \\
+ \sin n(\theta-\alpha)\big(-\sigma_1 A_1 d n K_{n,2} + \sigma_2 A_2 æ b_1 n K_{n,1}\big)\Big], \\
\sigma_{\zeta\eta,n}^{\Sigma L} = \frac{e_n}{J_0 \Omega_n p}\Big[\cos n(\theta-\alpha)\big(-\sigma_1 A_1 æ d \delta_2 K'_{n,2} + \sigma_2 A_2 b \delta_1 K'_{n,1}\big) \\
+ \sin n(\theta-\alpha)\big(\sigma_1 A_1 b n K_{n,2} + \sigma_2 A_2 æ d n K_{n,1}\big)\Big], \\
\sigma_{\eta\eta,n}^{\Sigma L} = \frac{e_n(1-\varepsilon)}{\Omega_n p}\Big[\cos n(\theta-\alpha)\sigma_1 A_1 \delta_2 K'_{n,2} - \sin n(\theta-\alpha)\sigma_2 A_2 n K_{n,1}\Big] - \sigma_{\zeta\zeta,n}^{\Sigma L}, \\
\Omega_n = -K_{n,1}K_{n,2}n^2 + K'_{n,1}K'_{n,2}\delta_1\delta_2.
\end{gathered} \tag{14}
$$

It seems impossible to represent expressions (14) in explicit form. Let us study the asymptotic behavior of the stresses at the cylinder surface a long time after the beginning of the process ($t \to \infty$), which corresponds to $p \to 0$ in the image space. It is shown in [18] that if the asymptotic behavior of the image has a singular point of an algebraic-logarithmic type as $p \to 0$:

$$
f^L(p) \sim -p^k/\ln p \qquad (k \neq 0, 1, 2, \ldots),
$$

then, as $t \to \infty$, the asymptotic behavior of the original has the form

$$
f(t) \sim \frac{t^{-k-1}}{(-k-1)!\ln t}.
$$

A plot of the function $f(t)$ at $k = -2$ is given in Fig. 2.

Assuming that p is small, we retain the first terms in the expansions of Bessel functions. The asymptotic behavior of the function $f(t)$ at $k = -2$ is used for $n = 1$. As a result, we obtain an approximate asymptotic representation for the total stresses at the surface of the rigid inclusion:

$$
\begin{gathered}
\sigma_{\zeta\zeta,0}^{\Sigma} = \big(\sigma_1 b_1 æ + \sigma_2 d\big)/J_0, \quad \sigma_{\zeta\eta,0}^{\Sigma} = \big(\sigma_1 d æ - \sigma_2 b\big)/J_0, \quad \sigma_{\eta\eta,0}^{\Sigma} = \big(\sigma_1 b_3 æ - \sigma_2 d\big)/J_0, \\
\sigma_{\zeta\zeta,1}^{\Sigma} = \frac{t}{\ln(t/\beta)}\frac{2c_2^2\delta}{J_0 r}\Big[\cos(\theta-\alpha)\big(\sigma_1 b_1 æ/c_1 + \sigma_2 d/c_2\big) + \sin(\theta-\alpha)\big(-\sigma_1 d/c_1 + \sigma_2 b_1 æ/c_2\big)\Big], \\
\sigma_{\zeta\eta,1}^{\Sigma} = \frac{t}{\ln(t/\beta)}\frac{2c_2^2\delta}{J_0 r}\Big[\cos(\theta-\alpha)\big(\sigma_1 d æ/c_1 - \sigma_2 b/c_2\big) + \sin(\theta-\alpha)\big(\sigma_1 b/c_1 + \sigma_2 d æ/c_2\big)\Big], \\
\sigma_{\eta\eta,1}^{\Sigma} = \frac{t}{\ln(t/\beta)}\frac{2c_2^2\delta}{J_0 r}\Big[\cos(\theta-\alpha)\big(\sigma_1 b_3 æ/c_1 - \sigma_2 d/c_2\big) + \sin(\theta-\alpha)\big(\sigma_1 d/c_1 + \sigma_2 b_3 æ/c_2\big)\Big], \\
\sigma_{\zeta\zeta,2}^{\Sigma} = 2\delta\Big[\cos 2(\theta-\alpha)\big(\sigma_1 b_1 æ^2 + \sigma_2 d\big) + \sin 2(\theta-\alpha)\big(-\sigma_1 d + \sigma_2 b_1\big)æ\Big]\Big/J_0, \\
\sigma_{\zeta\eta,2}^{\Sigma} = 2\delta\Big[\cos 2(\theta-\alpha)\big(\sigma_1 d æ^2 - \sigma_2 b\big) + \sin 2(\theta-\alpha)\big(\sigma_1 b + \sigma_2 d\big)æ\Big]\Big/J_0, \\
\sigma_{\eta\eta,2}^{\Sigma} = 2\delta\Big[\cos 2(\theta-\alpha)\big(\sigma_1 b_3 æ^2 - \sigma_2 d\big) + \sin 2(\theta-\alpha)\big(\sigma_1 d + \sigma_2 b_3\big)æ)\Big]\Big/J_0, \\
\sigma_{\zeta\zeta,n}^{\Sigma} = 0, \quad \sigma_{\zeta\eta,n}^{\Sigma} = 0, \quad \sigma_{\eta\eta,n}^{\Sigma} = 0 \quad (n \geqslant 3), \quad \beta = \frac{a_1 C}{2c_1 æ^{\delta/2}}, \quad \delta = \frac{1}{(1+æ)}, \quad b_3 = \frac{\varepsilon J_{,\zeta}^2 - J_{,\eta}^2}{æ}
\end{gathered} \tag{15}
$$

($C = 1.781072418\ldots$ is the Euler constant [17]). The asymptotic behavior of stresses in the reflected and diffraction waves has the form

$$\sigma^1_{\zeta\zeta,0} = -\sigma^1_{\eta\eta,0} = \big(\sigma_1 b\ae + \sigma_2 d\big)/J_0, \quad \sigma^1_{\zeta\eta,0} = \big(\sigma_1 d\ae - \sigma_2 b\big)/J_0, \quad \sigma^1_{\zeta\zeta,1} = \sigma^\Sigma_{\zeta\zeta,1}, \quad \sigma^1_{\zeta\eta,1} = \sigma^\Sigma_{\zeta\eta,1}, \quad \sigma^1_{\eta\eta,1} = \sigma^\Sigma_{\eta\eta,1},$$

$$\sigma^1_{\zeta\zeta,2} = (\ae - 1)\delta\Big[\cos 2(\theta-\alpha)\big(\sigma_1 b_4\ae + \sigma_2 d\big) + \sin 2(\theta-\alpha)\big(\sigma_1 d\ae - \sigma_2 b_4\big)\Big]\Big/J_0,$$

$$\sigma^1_{\zeta\eta,2} = (\ae - 1)\delta\Big[\cos 2(\theta-\alpha)\big(\sigma_1 d\ae + \sigma_2 b\big) + \sin 2(\theta-\alpha)\big(-\sigma_1 b\ae + \sigma_2 d\big)\Big]\Big/J_0, \tag{16}$$

$$\sigma^1_{\eta\eta,2} = (\ae - 1)\delta\Big[\cos 2(\theta-\alpha)\big(\sigma_1 b_5\ae + \sigma_2 d\big) + \sin 2(\theta-\alpha)\big(-\sigma_1 d\ae + \sigma_2 b_5\big)\Big]\Big/J_0,$$

$$b_4 = J^2_{,\zeta} + 3J^2_{,\eta}, \qquad b_5 = 3J^2_{,\zeta} + J^2_{,\eta}.$$

Let us consider a particular case. Let the ellipse tend to the circle ($a \to 0$, $\zeta_0 \to \infty$, $a \sinh\zeta_0 \to R$, $a\cosh\zeta_0 \to R$; R is the radius of the circle). Then from (15), we find an asymptotic solution to the problem of diffraction of elastic waves by a rigid round cylinder, coinciding with [16]:

$$\sigma^\Sigma_{rr,0} = -\sigma_1, \qquad \sigma^\Sigma_{r\alpha,0} = \sigma_2, \qquad \sigma^\Sigma_{\alpha\alpha,0} = -\varepsilon\sigma^\Sigma_{rr,0},$$

$$\sigma^\Sigma_{rr,1} = -\frac{t}{\ln(t/\beta)}\,\frac{2c_2^2\delta}{R}\left[\frac{\sigma_1}{c_1}\cos(\theta-\alpha) + \frac{\sigma_2}{c_2}\sin(\theta-\alpha)\right],$$

$$\sigma^\Sigma_{r\alpha,1} = \frac{t}{\ln(t/\beta)}\,\frac{2c_2^2\delta}{R}\left[\frac{\sigma_2}{c_2}\cos(\theta-\alpha) - \frac{\sigma_1}{c_1}\sin(\theta-\alpha)\right],$$

$$\sigma^\Sigma_{\alpha\alpha,1} = -\varepsilon\sigma^\Sigma_{rr,1}, \qquad \sigma^\Sigma_{rr,2} = -2\delta\big[\sigma_1\ae\cos 2(\theta-\alpha) + \sigma_2\sin 2(\theta-\alpha)\big],$$

$$\sigma^\Sigma_{r\alpha,2} = 2\delta\big[\sigma_2\cos 2(\theta-\alpha) - \sigma_1\ae\sin 2(\theta-\alpha)\big], \qquad \sigma^\Sigma_{\alpha\alpha,2} = -\varepsilon\sigma^\Sigma_{rr,2}.$$

Let us pass over to a finite-difference solution. We expand the potentials, by analogy with (12), into a Fourier series in terms of the angle $(\theta - \alpha)$

$$\varphi^1 = \sum_{n=0}^{\infty}\Big[\varphi^c_n\cos n(\theta-\alpha) + \varphi^s_n\sin n(\theta-\alpha)\Big], \quad \psi^1 = \sum_{n=0}^{\infty}\Big[\psi^c_n\cos n(\theta-\alpha) + \psi^s_n\sin n(\theta-\alpha)\Big]. \tag{17}$$

Then we have wave equations for each coefficient of the Fourier series (17):

$$\frac{\partial^2\varphi^{c,s}_n}{\partial t^2} = c_1^2\left(\frac{\partial^2\varphi^{c,s}_n}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi^{c,s}_n}{\partial r} - \frac{n^2}{r^2}\varphi^{c,s}_n\right), \quad \frac{\partial^2\psi^{c,s}_n}{\partial t^2} = c_2^2\left(\frac{\partial^2\psi^{c,s}_n}{\partial r^2} + \frac{1}{r}\frac{\partial\psi^{c,s}_n}{\partial r} - \frac{n^2}{r^2}\psi^{c,s}_n\right). \tag{18}$$

After expansion into a Fourier series, the boundary conditions (3) will be the following:

$$\varphi^c_{n,r} = J_0^{-1/2}\big(J_{,\eta}u^{0c}_{\eta,n} - J_{,\zeta}u^{0c}_{\zeta,n}\big) + n\psi^s_n/r, \qquad \varphi^s_{n,r} = J_0^{-1/2}\big(J_{,\eta}u^{0s}_{\eta,n} - J_{,\zeta}u^{0s}_{\zeta,n}\big) - n\psi^c_n/r,$$

$$\psi^c_{n,r} = J_0^{-1/2}\big(J_{,\eta}u^{0c}_{\zeta,n} + J_{,\zeta}u^{0c}_{\eta,n}\big) - n\varphi^s_n/r, \qquad \psi^s_{n,r} = J_0^{-1/2}\big(J_{,\eta}u^{0s}_{\zeta,n} + J_{,\zeta}u^{0s}_{\eta,n}\big) + n\varphi^c_n/r \tag{19}$$

$$\left(r = a\sqrt{\sinh^2\zeta_0 + \cos^2\eta}\right).$$

Here $u^{0s}_{\zeta,n}$, $u^{0s}_{\eta,n}$, $u^{0c}_{\zeta,n}$, $u^{0c}_{\eta,n}$ are the Fourier series coefficients for the displacements in the incident wave. The system of equations (18) with boundary conditions (19) is solved by the finite difference method using an explicit scheme of the "cross" type. The steps of the difference grid are chosen under the Courant stability condition. In order to minimize the numerical variance, we assume that $c_1\tau = h_\varphi$, $c_2\tau = h_\psi$, where τ is the time step, h_φ, h_ψ are the steps in space for the equations for the scalar and vector potentials of displacements, respectively.

The major semi-axis of the ellipse ($a_1 = a\cosh\zeta_0$), propagation speed of longitudinal waves c_1, and the density of the elastic medium ρ are taken as the measurement units of the distance, speed, and density.

Figures 3–19 give plots of the distribution of stresses with the angle η at the moment of time $t = 10$ and oscillograms of the stresses. The stresses were calculated in the reflected and diffraction waves at $\theta = 0$,

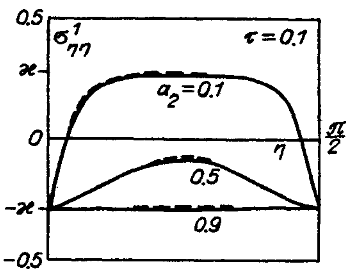

Fig. 3

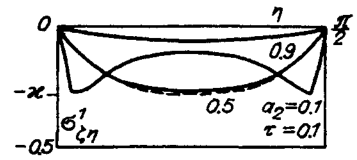

Fig. 4

$\nu = 0.3$. Different ratios of the ellipse axes were considered: $a_2 = 0.1$, 0.5, 0.9 ($a_2 = a \sinh \zeta_0$ is the minor semi-axis of the ellipse). The solid curves correspond to the finite-difference solution. and the dashed curves, to the asymptotic solution (16).

Figures 3–11 present plots of the stresses occurring under the action of the longitudinal wave ($\sigma_1 = -1$, $\sigma_2 = 0$). The calculation results at $n = 0$ are given in Figs. 3–6. The calculations were made at $\tau = 0.1$. It is seen that at $n = 0$, beginning from $t \geqslant 10$, the stresses coincide with the asymptotic solution (16) to the accuracy of plot drawing errors. The analysis of formulas (16) and Figs. 3–6 show that the following inequalities are satisfied for the incident longitudinal wave:

$$-æ \leqslant \sigma^1_{\zeta\zeta}/\sigma_1 \leqslant æ, \qquad -æ \leqslant \sigma^1_{\eta\eta}/\sigma_1 \leqslant æ, \qquad 0 \leqslant \sigma^1_{\zeta\eta}/\sigma_1 \leqslant æ.$$

Figures 7–9 illustrate the behavior of stresses at $n = 1$. As the calculations showed, at $a_2 = 0.9$, beginning from $t \simeq 5$, the finite-difference solution and the asymptotics coincide. If the minor semi-axis of the ellipse becomes smaller, the agreement between the asymptotics and the numerical calculation is achieved later. In particular, at $a_2 = 0.1$ this is achieved even for $t \simeq 10$ (Fig. 9). As a_2 decreases, one has to refine considerably the steps of the difference grid, in order to get the required accuracy in the finite-difference solution of the problem. For $a_2 = 0.9$ it is sufficient to assume that $\tau = 0.02$, whereas for $a_2 = 0.1$ the step $\tau = 0.005$ does not permit computation to an acceptable accuracy in all the angle points.

The calculation results at $n = 2$ are presented in Figs. 10 and 11. It is seen from comparison of the asymptotic and numerical solutions that the values of stresses at $t > 2$ remain constant in time and tend to the asymptotic values (16) as the steps of the difference grid are decreased. The calculations made at $n = 3$, 4, 5 show that the amplitudes of perturbations approach zero at $t \simeq 4$.

In the oscillograms of stresses (Figs. 5, 6, 9–11), the time of occurrence of perturbations corresponds to the time of arrival, at the given point r, of an axisymmetric longitudinal wave beginning to move at $t = 0$ from the radius $r = a_1$ to the center of the ellipse. This time is determined from the formula $t_1 = (a_1 - r)/c_1$. The time of termination of the sharp peaks of perturbations corresponds to the arrival of the axisymmetric wave "reflected from the center" and is given by the formula $t_2 = (a_1 + r)/c_1$.

Figures 12–19 show the stresses calculated under the action of a shear wave ($\sigma_1 = 0$, $\sigma_2 = -1$), Figs. 12–14 show the zero form, Figs. 15–19, the first form, and Figs. 18, 19, the second form. As in the case of action of a longitudinal wave, the oscillograms of stresses at $t \simeq 10/c_1$ practically coincide with the asymptotic solution for $n = 0$, 1, 2. The time of appearance of perturbations and the time of termination of sharp peaks of perturbations in oscillograms are determined from formulas similar to the case of a longitudinal wave, in which c_2 is used instead of c_1:

$$t_1 = (a_1 - r)/c_2, \qquad t_2 = (a_1 + r)/c_2.$$

Thus, the comparison of numerical and analytical solutions shows that at $t \simeq 10/c_1$ the parameters of perturbations coincide with the asymptotic solution to high accuracy (16).

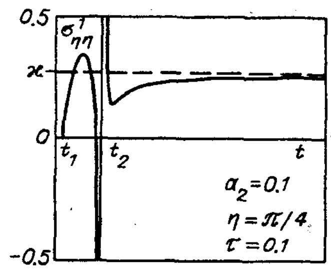

Fig. 5

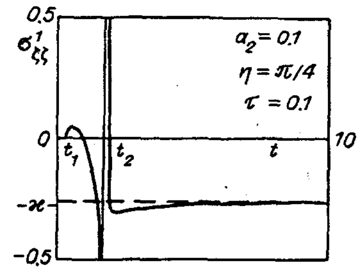

Fig. 6

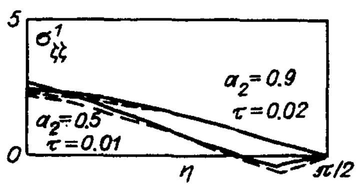

Fig. 7

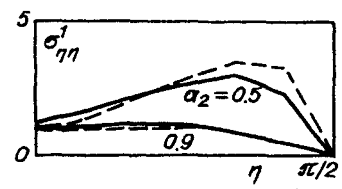

Fig. 8

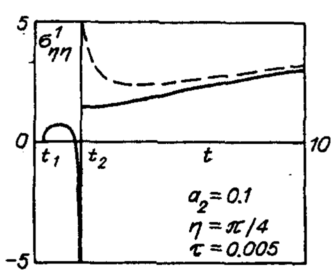

Fig. 9

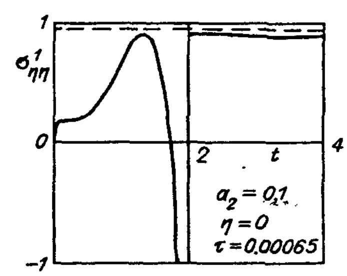

Fig. 10

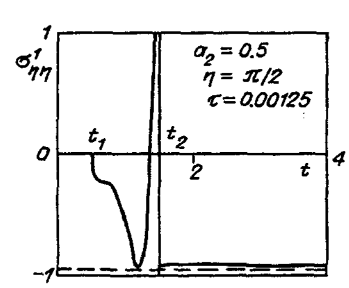

Fig. 11

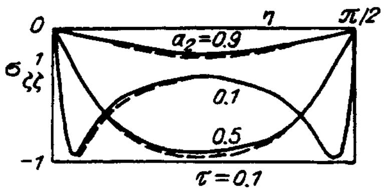

Fig. 12

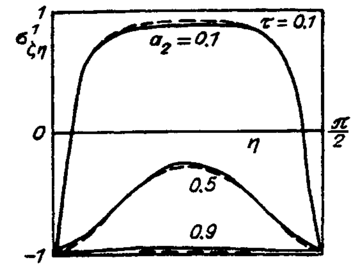

Fig. 13

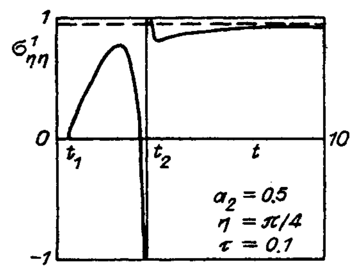

Fig. 14

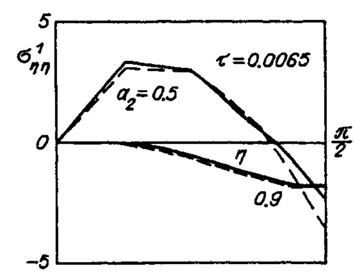

Fig. 15

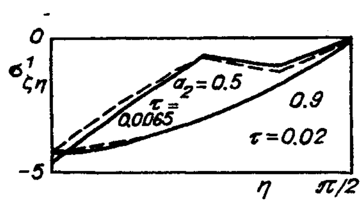

Fig. 16

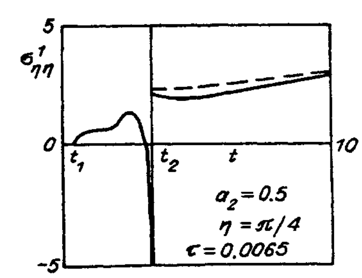

Fig. 17

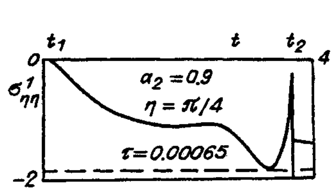

Fig. 18

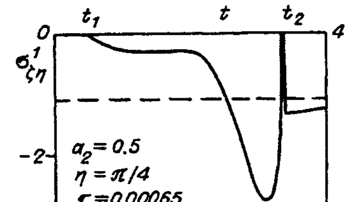

Fig. 19

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